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Addition of arbitrary number of identical angular momenta

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Abstract. Under the vector addition of angular momenta, each equal to s, the resulting angular momenta j occur with some multiplicity. In this paper the general formula for these multiplicities is obtained by a simple algebraic method. In the case of $s = 1/2$ our expression proves to be a well known one. Also given are some relations connected with the multiplicities.

1. Introduction

There has been appreciable success in the theory of cooperative effects of many atoms interacting with electromagnetic radiation. Particular attention was drawn to the case of two-level atoms. That is quite understandable because of its simplicity, completeness of theoretical results and paramount practical importance. Some of the papers in this field such as the basic paper of Dicke (1954), the first paper on atomic coherent states (Arecchi *et al* 1972) and the first paper on a theoretical investigation of super-radiant phase transition in the Dicke model (Hepp and Lieb 1973) essentially use the formula

$$P_{jn} = \frac{n!(2j+1)}{(\frac{1}{2}n - j)!(\frac{1}{2}n + j + 1)!} \tag{1}$$

which gives us the expression for multiplicity P_{jn} of angular momentum j occurring in quantum mechanical vector addition of n identical spins, each having $s = 1/2$. Formula (1) was also derived by Kotani *et al* (1955) in the genealogical construction for electronic spin states. In this construction P_{jn} are the numbers of spin functions encountered in the so called Yamanuchi–Kotani branching diagram (Salmon 1974).

Long before these works (1) was known in symmetric group theory as a particular case of the Frobenius formula (Hamermesh 1962) for the dimensions of the irreducible representations, corresponding to the frame $\{\lambda_1, \lambda_2\}$. But it is well known that the dimension of symmetric group representation, corresponding to the Young tableaux with three and more rows, is not equal to the multiplicity P_{jn} of angular momentum j occurring in vector addition of n identical spins, each having the angular momentum $s \geqslant 1$.

The purpose of this paper is to find the formula (10) generalizing (1) to the case of addition of n identical angular momenta each of which is equal to s. The explicit form of the new formula turned out to be much more complicated than that of (1) and it is equal to the sum of $k = [(sn - j)/(2s+1)]$ terms. It will be shown that if we replace s in (10) by $1/2$ we get (1) exactly. After that we will give the proof of this formula in an arbitrary case and some useful relations connected with it.

2. General expressions for the multiplicities

If we have particles, one in the state with angular momentum j_1 and the other with angular momentum j_2, the angular momentum of the two-particle system j will satisfy the well known restriction

$$j_1+j_2 \geqslant j \geqslant |j_1-j_2|. \tag{2}$$

In the present paper we will dicuss only the case of identical angular momenta (or spins) i.e. $j_1=j_2=\ldots=j_n=s$, and we will denote by j the total angular momentum of the system. Then (2) can be rewritten

$$2s \geqslant j \geqslant 0. \tag{3}$$

Let us consider a three-particle system. After two-step addition of three angular momenta we obtain the following conditions:

$$\begin{aligned} j = {} & 3s, 3s-1, \ldots s, \\ & 3s-1, 3s-2, \ldots s-1, \\ & \ldots . \end{aligned} \tag{4}$$

We can see that some values of j occur in (4) several times, otherwise multiplicities of such values are more than one. In general, after the addition of n spins, each having the angular momentum s, definite j occur P^s_{jn} times (0 or $1/2 \leqslant j \leqslant sn$). We can paraphrase this in group theory language: the direct product of n irreducible representations D^s of SU(2) decomposes into irreducible ones D^j, and P^s_{jn} is the multiplicity of the representation D^j.

Since each particle of the system may be in one of $2s+1$ independent states and the whole system in one of $(2s+1)^n$ states, equality of state numbers in the system of independent angular momenta and in the system of coupling angular momenta may be written in the form:

$$\sum_j (2j+1)P^s_{jn} = \chi^n, \tag{5}$$

where $j=sn, sn-1, \ldots 0$ or $1/2$, and $\chi = 2s+1$.

For $s=1/2$ the multiplicities P^s_{jn} are given in (1). There is no such simple formula for the case $s \geqslant 1$. But as we will see later it is possible to write the numbers P^s_{jn} in an analytical expression and our aim is to find this.

Tables 1 and 2 of numbers P^s_{jn}, which are an equivalent Yamanuchi–Kotani branching diagram for $s>1/2$, may serve as illustrations of our reasonings.

Referring to table 1 it can be easily shown that P^1_{jn} are governed by such recurrence relations as:

$$\begin{aligned} & P^1_{jn} = P^1_{j+1,n-1} + P^1_{j,n-1} + P^1_{j-1,n-1}, \qquad j=n, n-1, \ldots 1, \\ & P^1_{0n} = P^1_{1,n-1}. \end{aligned} \tag{6}$$

For arbitrary s the general recurrence relations then take the form

$$P^s_{jn} = P^s_{j+s,n-1} + P^s_{j+s-1,n-1} + \ldots + P^s_{|j-s|,n-1}. \tag{7}$$

It is not possible to obtain the solution of the most simple relation (6) by a straightforward method. In order to find the solution of (6) we draw attention to the simplicity

Table 1. Multiplicities P^1_{jn}.

$j \backslash n$	0	1	2	3	4	5	6	7	8
0	1	0	1	1	3	6	15	36	91
1		1	1	3	6	15	36	91	232
2			1	2	6	15	40	105	280
3				1	3	10	29	84	238
4					1	4	15	49	154
5						1	5	21	76
6							1	6	28
7								1	7
8									1

of the numbers filling the first few lower diagonals in table 1:

$$\begin{aligned}
P^1_{n,n} &= \binom{n-2}{0} = 1,\\
P^1_{n-1,n} &= \binom{n-1}{1} = n-1,\\
P^1_{n-2,n} &= \binom{n}{2},\\
P^1_{n-3,n} &= \binom{n+1}{3} - n\binom{n-2}{0},\\
P^1_{n-4,n} &= \binom{n+2}{4} - n\binom{n-1}{1},\\
P^1_{n-5,n} &= \binom{n+3}{5} - n\binom{n}{2},\\
P^1_{n-6,n} &= \binom{n+4}{6} - n\binom{n+1}{3} + \binom{n}{2}\binom{n-2}{0}.
\end{aligned} \tag{8}$$

Here $\binom{m}{n}$ are the binomial coefficients. Now we have enough information to understand how to write all P^1_{jn} in the unique expression

$$P^1_{jn} = \sum_k (-1)^k \binom{n}{k} \binom{2n-3k-j-2}{n-2}. \tag{9}$$

Here $0 \leqslant k \leqslant [(n-j)/3]$. The square brackets $[b]$ denote the integer part of b. Extending the construction in a similar fashion, we achieve the formula for arbitrary s:

$$P^s_{jn} = \sum_k (-1)^k \binom{n}{k} \binom{(s+1)n-j-\chi k-2}{n-2}. \tag{10}$$

Here and later in all summations over k numbers, k must satisfy two conditions: (i) $k \geqslant 0$ and (ii) the upper numbers in binomial coefficients must be more than or equal to

Table 2. Multiplicities $P_{jn}^{3/2}$.

j \ n	0	1	2	3	4	5	6	7	8
0	1		1		4		34		364
		0		2		20		210	
1			1		9		90		1000
		1		4		34		364	
2			1		11		120		1400
				3		36		426	
3			1		10		120		1505
				2		30		400	
4					6		96		1351
				1		20		315	
5					3		64		1044
						10		210	
6					1		35		700
						4		119	
7							15		406
						1		56	
8							5		202
								21	
9							1		84
								6	
10									28
								1	
11									7
12									1

the lower numbers. If $s = 1/2$ is inserted, (10) must be in agreement with (1). In fact, from (10) we have

$$P_{jn} = \sum_k (-1)^k \binom{n}{k} \binom{\frac{3}{2}n - j - 2k - 2}{n-2}. \tag{11}$$

In elementary combinatorics (Vilenkin 1969) there are the following relations (written here in a convenient form):

$$Q_{jn} = \sum_k (-1)^k \binom{n}{k} \binom{\frac{3}{2}n - j - 2k - 1}{n-1} = \binom{n}{\frac{1}{2}n - j}. \tag{12}$$

The middle part of (12) is almost equal to (11). Since $P_{jn} = Q_{jn} - Q_{j+1,n}$ it follows that

$$P_{jn} = \binom{n}{\frac{1}{2}n - j} - \binom{n}{\frac{1}{2}n - j - 1} = \frac{n!(2j+1)}{(\frac{1}{2}n - j)!(\frac{1}{2}n + j + 1)!} \tag{13}$$

which was to be proved.

Further we will write (10) in the cases $s = 1, j = n - 3$ and $s = \frac{3}{2}, j = \frac{3}{2}n - 4$. These are the simplest expressions for $s = 1$ and $s = \frac{3}{2}$ which have more than one term

$$P^1_{n-3,n} = n(n^2 - 7)/6,$$
$$P^{\frac{3}{2}}_{\frac{3}{2}n-4,n} = n(n^3 - 2n^2 - n - 26)/24.$$

The algebraic equations $n^2 - 7 = 0$ and $n^3 - 2n^2 - n - 26 = 0$ have no integer roots.

Therefore these multiplicities cannot be written in the expression analogous to (1). Comparable results give considerations of multiplicities with another s and j. Consequently we may say that in general (10) is not reducible to the monomial form akin to (1).

3. Recurrence relations for multiplicities

We will now give the proofs that the multiplicities (10) satisfy the recurrence relations (7) and therefore are the true multiplicities.

Suppose $j \geqslant s$; then from (7) we obtain

$$R_{jn} = P_{jn} - \sum_{i=-s}^{s} P_{j+i,n-1} = 0. \tag{14}$$

For the sake of brevity, the index s in R and P is omitted. The substitution of P_{jn} from (10) into (14) yields

$$R_{jn} = \sum_k (-1)^k \binom{n}{k}\binom{m-\chi k-2}{n-2} - \sum_{l=0}^{2s}\sum_k (-1)^k \binom{n-1}{k}\binom{m-l-\chi k-3}{n-3} \tag{15}$$

where $m=(s+1)n-j$, $l=s+i$. Since the maximum value of k is $p=(sn-j-l_0)/\chi$, where $0 \leqslant l_0 \leqslant 2s$, we present a more simple form of (15):

$$R_{jn} = \sum_{k=0}^{p-1}(-1)^k F_{jn}^k + (-1)^p\binom{n}{p}\binom{m-\chi p-2}{n-2} - \sum_{l=0}^{l_0}(-1)^p\binom{n-1}{p}\binom{m-l-\chi p-3}{n-3} \tag{16}$$

where

$$F_{jn}^k = \binom{n}{k}\binom{m-\chi k-2}{n-2} - \sum_{l=0}^{2s}\binom{n-1}{k}\binom{m-l-\chi k-3}{n-3} \tag{17}$$

$$= \binom{n-1}{k-1}\binom{m-\chi k-2}{n-2} + \binom{n-1}{k}\binom{m-\chi(k+1)-2}{n-2}. \tag{18}$$

To make the transformation from (17) to (18) we used the following binomial relations:

$$\sum_{l=0}^{k}\binom{m-l}{n} = \binom{m+1}{n+1} - \binom{m-k}{n+1}. \tag{19}$$

Here in case $k=m-n$ the last term is equal to $\binom{n}{n+1}$ and is nothing but zero. It is not difficult to show that

$$\sum_{k=0}^{t}(-1)^k F_{jn}^k = (-1)^t\binom{n-1}{t}\binom{m-\chi(t+1)-2}{n-2}. \tag{20}$$

The two last terms of (16) can be simplified with the help of (19), so that R_{jn} is given by:

$$R_{jn} = (-1)^p\binom{n-1}{p}\binom{m-l_0-\chi p-3}{n-2}$$

and since we have the condition $p\chi = sn-j-l_0$ and restriction $\binom{n-3}{n-2} \equiv 0$, then

$$R_{jn} = (-1)^p\binom{n-1}{p}\binom{n-3}{n-2} \equiv 0. \tag{21}$$

We have proved that the numbers P_{jn} from (10) satisfy recurrence relations (7) when $j \geqslant s$.

When $j \leqslant s$ (7) is given by

$$R_{jn} = P_{jn} - \sum_{i=-j}^{j} P_{s+i,n-1}. \tag{22}$$

The substitution of (10) into (22) yields

$$R_{jn} = \sum_k (-1)^k \binom{n}{k} \binom{m - \chi k - 2}{n-2} - \sum_{l=0}^{2j} \sum_k (-1)^k \binom{n-1}{k} \binom{m + 2j - l - \chi k - 2}{n-3} \tag{23}$$

where $l = j + i$, $m = (s+1)n - j$. After the summation over l with the help of (19) the last binomial coefficient in the second term of (23) is

$$\binom{m + 2j - \chi(k+1) - 1}{n-2} - \binom{m - \chi(k+1) - 2}{n-2}.$$

Substituting this into (23) we arrive at three sums over k. The first and the third sums we can put together. After that we have

$$R_{jn} = \sum_k (-1)^k \binom{n-1}{k} \left[\binom{m - \chi k - 2}{n-2} - \binom{m - 2(s-j) - \chi k - 2}{n-2} \right] = 0. \tag{24}$$

R_{jn} is identical with zero when $j = s$. However we failed to make the general proof of (24) for $j < s$. We performed the verification of (24) for some concrete values of s, n and j which confirmed the correctness of this equality and therefore of (22).

4. Some properties of multiplicities

Getting the evidence of rightness of the expression (10) for the multiplicities we can derive several formulae from general relation (5). They may be useful in statistical investigation of many-spin systems.

Substituting (10) into (5) and changing the sequence of summation over j and k we have

$$\sum_k (-1)^k \binom{n}{k} \sum_j (2j+1) \binom{q + l - j}{l} = \chi^n, \tag{25}$$

$$0 \leqslant k \leqslant \left[\frac{sn}{\chi}\right],\ j = q, q-1, \ldots, 0 \text{ or } \tfrac{1}{2}, \qquad q = sn - \chi k,\ l = n-2.$$

For binomial coefficients there are two simple formulae which are subsidiary for us

$$\sum_{i=0}^{m} \binom{n+i}{n} = \binom{m+n+1}{n+1}, \tag{26}$$

$$\sum_{i=0}^{m} i \binom{n+i}{n} = (n+1) \binom{m+n+1}{n+2}. \tag{27}$$

They can be easily proved by elementary methods. With the help of (26) and (27) we can take the sum (25) over j

$$\sum_j (2j+1)\binom{q+l-j}{l} = \begin{cases} \dfrac{l+2q+2}{l+2}\dbinom{q+l+1}{l+1}, & j=0, 1, \ldots, m \\ \dfrac{2l+2q+3}{l+2}\dbinom{l+q+\frac{1}{2}}{l+1}, & j=\frac{1}{2}, \frac{3}{2}, \ldots, m. \end{cases} \tag{28}$$

Finally we have from (25)

$$\sum_k (-1)^k \frac{n-2k}{n}\binom{n}{k}\binom{(s+1)n-\chi k-1}{n-1} = \chi^{n-1}, \qquad sn \text{ integral} \tag{29}$$

$$2\sum_k (-1)^k \binom{n}{k}\binom{(s+1)n-\chi k-\frac{1}{2}}{n} = \chi^n, \qquad sn \text{ half-integral.} \tag{30}$$

Using the resembling methods it is possible to take other sums

$$\sum_{j=0}^{sn} P^s_{jn} = \sum_k (-1)^k \binom{n}{k}\binom{(s+1)n-\chi k-1}{n-1}, \qquad sn \text{ integral,} \tag{31}$$

$$\sum_{j=1/2}^{sn} P^s_{jn} = \sum_k (-1)^k \binom{n}{k}\binom{(s+1)n-\chi k-3/2}{n-1}, \qquad sn \text{ half-integral.} \tag{32}$$

With the help of the last expressions and formula (5) it is not difficult to take the sum $\sum_j jP^s_{jn}$.

In conclusion, it is interesting to note that in our problem there are some sorts of numbers which are determined similarly to (10) and which include the multiplicities P^s_{jn} as a particular case. We now speak about the differences

$$P^{s\nu}_{jn} = P^{s\nu}_{jn} - P^{s,\nu-1}_{j+1,n}, \qquad P^{s,0}_{jn} \equiv P^s_{jn}.$$

By means of (19), where we set $k=0$, one obtains

$$P^{s,1}_{jn} = P^s_{jn} - P^s_{j+1,n} = \sum_k (-1)^k \binom{n}{k}\binom{(s+1)n-\chi k-j-3}{n-3}.$$

After the analogous continuation of calculation we have for arbitrary ν

$$P^{s\nu}_{jn} = \sum_k (-1)^k \binom{n}{k}\binom{(s+1)n-\chi k-j-\nu-2}{n-\nu-2}.$$

References

Arecchi F T, Courtens E, Gilmore R and Thomas H 1972 *Phys. Rev.* A **6** 2211–37
Dicke R 1954 *Phys. Rev.* **93** 99–110
Hamermesh M 1962 *Group Theory* (Reading, Mass.: Addison-Wesley)
Hepp K and Lieb E H 1973 *Ann. Phys., NY* **76** 360–404
Kotani M, Amemiya A, Ishiguro E and Kimura T 1955 *Tables of Molecular Integrals* (Tokyo: Maruzen)
Salmon W 1974 *Adv. Quantum Chem.* **8** 37–94
Vilenkin N Ya 1969 *Kombinatorika* (Moskva: Nauka)